

MOTIVIC CLASSES OF COMMUTING VARIETIES VIA POWER STRUCTURES

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ABSTRACT. We prove a formula, originally due to Feit and Fine, for the class of the commuting variety in the Grothendieck group of varieties. Our method, which uses a power structure on the Grothendieck group of stacks, allows us to prove several refinements and generalizations of the Feit-Fine formula. Our main application is to motivic Donaldson-Thomas theory.

1. INTRODUCTION

Let $K_0(Var_{\mathbb{C}})$ be the Grothendieck group of varieties over \mathbb{C} , i.e. the free Abelian group generated by isomorphism classes of varieties over \mathbb{C} with the relation

$$[V] = [V - Z] + [Z]$$

for closed subvarieties $Z \subset V$. Cartesian product induces a ring structure on $K_0(Var_{\mathbb{C}})$. We refer to the class $[V]$ of a variety V as the *motivic class* of V .

Let $C(n)$ be the variety of commuting n by n matrices:

$$C(n) = \{ (A, B) \in \text{End}(n)^2 : [A, B] = 0 \}.$$

The motivic class of $C(n)$ is given by a formula which is essentially due to Feit and Fine [5]¹:

$$(1) \quad [C(n)] = [\text{GL}(n)] \sum_{\alpha \vdash n} \prod_{k=1}^{\infty} \frac{[\text{End}(b_k(\alpha))]}{[\text{GL}(b_k(\alpha))]} [\mathbb{A}_{\mathbb{C}}^{b_k(\alpha)}]$$

where the sum is over partitions of n with the notation that $b_k(\alpha)$ is the number of parts of size k in α . The motivic class of $C(n)$ lies in the subring given by polynomials in the Lefschetz motive $\mathbb{L} := [\mathbb{A}_{\mathbb{C}}^1]$. This follows easily from equation (1) using the elementary formula

$$[\text{GL}(b)] = (\mathbb{L}^b - 1)(\mathbb{L}^b - \mathbb{L}) \dots (\mathbb{L}^b - \mathbb{L}^{b-1}).$$

In this paper, we give a new proof of Equation (1) using power structures, and we prove several refinements and generalizations.

For example, we prove that each summand in Equation (1) has a geometric interpretation. Let α be a partition of n . We say that $A \in \text{End}(n)$ has *Jordan type* α if the Jordan normal form of A has $b_k(\alpha)$ blocks of size k for all k . We define

$$C(\alpha) = \{ (A, B) \in \text{End}(n)^2 : [A, B] = 0, A \text{ has Jordan type } \alpha \}.$$

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¹Feit and Fine were counting points in $C(n)$ over the finite field \mathbb{F}_q , but their method can be made to work in the motivic setting.

Theorem 1. *The motivic class of $C(\alpha)$ in $K_0(\text{Var}_{\mathbb{C}})$ is given by*

$$(2) \quad [C(\alpha)] = [\text{GL}(n)] \prod_{k=1}^{\infty} \frac{[\text{End}(b_k(\alpha))]}{[\text{GL}(b_k(\alpha))]} \mathbb{L}^{b_k(\alpha)}$$

This theorem can be concisely expressed in terms of a generating function. Let $M_{\mathbb{C}}$ be the Grothendieck group of varieties localized at the classes of the general linear groups, namely

$$M_{\mathbb{C}} = K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}, (1 - \mathbb{L}^{-b})^{-1}, b > 0].$$

We can regard $M_{\mathbb{C}}$ as a subring of $K_0(\text{Var}_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ via² the geometric series expansion of $(1 - \mathbb{L}^{-b})^{-1}$.

Theorem 1'. *The following equation holds in $M_{\mathbb{C}}[[t_1, t_2, \dots]]$:*

$$(3) \quad \sum_{\alpha} \frac{[C(\alpha)]}{[\text{GL}(|\alpha|)]} \prod_{k=1}^{\infty} t_k^{b_k(\alpha)} = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 - \mathbb{L}^{2-m} t_k)^{-1}.$$

In particular, by setting $t_k = t^k$, we get

$$\sum_{n=0}^{\infty} \frac{[C(n)]}{[\text{GL}(n)]} t^n = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 - \mathbb{L}^{2-m} t^k)^{-1}.$$

Theorems 1 and 1' are equivalent: Equation (3) is obtained from Equation (2) by multiplying by $\prod_k t_k^{b_k(\alpha)} / [\text{GL}(|\alpha|)]$, summing over partitions, reversing the order of the sum and the product, and then applying Euler's formula (Equation (10)).

Our other generalization concerns a version of the commuting variety where the matrices are required to have a certain block form. Let

$$V = V_1 \oplus \dots \oplus V_r$$

where V_k is a complex vector space of dimension n_k . Suppose that $A, B \in \text{End}(V)$ are of the form

$$\begin{aligned} A &= A_1 \oplus \dots \oplus A_r & A_k &\in \text{Hom}(V_k, V_{k+1}) \\ B &= B_1 \oplus \dots \oplus B_r & B_k &\in \text{Hom}(V_k, V_k) \end{aligned}$$

where we regard the indices as taking values in \mathbb{Z}/r . We say that A is of *cyclic block type* and we say that B is of *diagonal block type*. Matrices of this type determine a representation of the quiver given in figure 1.

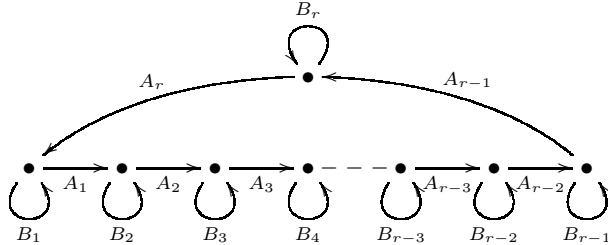


FIGURE 1. Quiver associated to the matrices from Theorem 2.

²By $K_0(\text{Var}_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ we really mean the completion of $K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ in the dimension filtration (see for example [2, § 2]). Laurent series in \mathbb{L}^{-1} make sense in this ring.

Let $\underline{n} = (n_1, \dots, n_r)$ and consider the variety of commuting pairs of endomorphisms in the above form:

$$C(\underline{n}) = \{(A, B) \in \text{End}(V)^2 \text{ in the above form and } [A, B] = 0\}.$$

Theorem 2. *Let $C(\underline{n})$ be as above and let $G(\underline{n}) = \text{GL}(V_1) \times \dots \times \text{GL}(V_r)$. Then*

$$\sum_{\underline{n}} \frac{[C(\underline{n})]}{[G(\underline{n})]} t_1^{n_1} \dots t_r^{n_r} = \prod_{m=1}^{\infty} (1 - \mathbb{L} t^m)^{-1} \prod_{k=0}^{\infty} \prod_{(a,b)} (1 - \mathbb{L}^{-k} t_{[a,b]} t^{m-1})^{-1}$$

where $t = t_1 \dots t_r$, $t_{[a,b]} = t_a t_{a+1} \dots t_b$, and the last product runs over all $(a, b) \in (\mathbb{Z}/r)^2$.

Remark 3. Theorems 1' and 2 have been used to compute motivic Donaldson-Thomas invariants. The Feit-Fine formula was used in [1] to compute the motivic DT invariants of \mathbb{C}^3 while Theorem 2 was used in conjunction with the dimensional reduction technique of [11] to compute the motivic DT invariants of the orbifold $\mathbb{C}^3/(\mathbb{Z}/r)$ (see [11, § 8]).

2. POWER STRUCTURES ON THE GROTHENDIECK GROUPS OF VARIETIES AND STACKS

The key technical tool we use are the power structures on the Grothendieck groups of varieties and stacks. The notion of a power structure is due to Gusein-Zade, Luengo, and Melle-Hernandez [8]. It is closely related to pre-lambda ring structures, although we will not use that language here. In this section we describe the power structures on the Grothendieck groups and we explain a geometric interpretation which is of key importance to us.

Definition 4 ([8]). Let R be a commutative ring with identity. A *power structure* is a map $(1 + tR[[t]]) \times R \rightarrow 1 + tR[[t]]$, denoted $(A(t), M) \mapsto A(t)^M$ taking a pair $(A(t), M)$ consisting of $A(t)$, a formal power series in t with coefficients in R having constant term 1, and $M \in R$ to a formal power series $A(t)^M$ satisfying the following expected properties of exponentiation:

- (1) $A(t)^0 = 1$
- (2) $A(t)^1 = A(t)$
- (3) $(A(t) \cdot B(t))^M = A(t)^M \cdot B(t)^M$
- (4) $A(t)^{M+N} = A(t)^M \cdot A(t)^N$
- (5) $A(t)^{M \cdot N} = (A(t)^M)^N$
- (6) $(1 + t)^M = 1 + Mt + O(t^2)$
- (7) $A(t^k)^M = (A(t))^M|_{t \mapsto t^k}$.

The Grothendieck group of varieties $K_0(\text{Var}_{\mathbb{C}})$ has a natural power structure uniquely determined by the equation

$$\left(\sum_{k=0}^{\infty} t^k \right)^{[X]} = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n$$

where X is a variety and $\text{Sym}^n X$ is the n th symmetric product of X . This power structure is *effective* in the sense that it respects the semi-ring $S_0(\text{Var}_{\mathbb{C}}) \subset K_0(\text{Var}_{\mathbb{C}})$ of effective classes, i.e. those represented by the classes of varieties. Moreover, on effective classes, $A(t)^{[M]}$ has an important geometric interpretation due to Gusein-Zade, Luengo, and Melle-Hernandez.

Suppose that M is a variety and

$$A(t) = \sum_{k=0}^{\infty} [A_k] t^k$$

where A_k are varieties with A_0 a point. Then the coefficient of t^n in $A(t)^{[M]}$ is given by the class of the variety

$$(4) \quad B_n = \bigsqcup_{\alpha \vdash n} \left(\prod_{i=1}^{\infty} M^{b_i(\alpha)} \setminus \Delta \right) \times_{S_\alpha} \left(\prod_{i=1}^{\infty} A_i^{b_i(\alpha)} \right)$$

where $b_i(\alpha)$ is the number of parts of size i in the partition α , Δ is the large diagonal in $\prod_{i=1}^{\infty} M^{b_i(\alpha)}$, and S_α is the product $\prod_{i=1}^{\infty} S_{b_i(\alpha)}$ of symmetric groups acting on each factor in the obvious way. The variety B_n has the following geometric interpretation; it parameterizes maps

$$(5) \quad \phi : S \rightarrow \bigsqcup_{i=1}^{\infty} A_i$$

where $S \subset M$ is a finite subset and ϕ satisfies

$$n = \sum_{x \in S} i(\phi(x))$$

where $i : \sqcup_i A_i \rightarrow \mathbb{N}$ is the tautological map taking the value i on the component A_i . We think of B_n as parameterizing a finite set of “particles” on M of total “charge” n where the “internal state space” of a charge i particle is A_i .

The power structure on $K_0(Var_{\mathbb{C}})$ and the above geometric interpretation have an extension to $K_0(Stck_{\mathbb{C}})$, the Grothendieck group of stacks having affine stabilizer groups:

It is shown in [4] that $K_0(Stck_{\mathbb{C}})$ is isomorphic to $M_{\mathbb{C}} = K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}, (1 - \mathbb{L}^n)^{-1}]$, the Grothendieck group of varieties localized at the classes \mathbb{L} and $(1 - \mathbb{L}^n)$ for $n \geq 1$ (this is equivalent to localizing at the classes $[GL_n]$ for all n). Via the expansion of the classes $(1 - \mathbb{L}^n)^{-1}$ as Laurent series in \mathbb{L}^{-1} , there is a map

$$K_0(Stck_{\mathbb{C}}) \rightarrow K_0(Var_{\mathbb{C}})[[\mathbb{L}^{-1}]]$$

to the completion of $K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ in the dimension filtration (which we denote somewhat abusively by $K_0(Var_{\mathbb{C}})[[\mathbb{L}^{-1}]]$).

The power structure on $K_0(Var_{\mathbb{C}})$ has a unique extension to $K_0(Stck_{\mathbb{C}})$ and $K_0(Var_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ characterized by the property

$$(6) \quad (1 - t)^{-\mathbb{L}^k[M]} = (1 - \mathbb{L}^k t)^{-[M]}$$

for all $k \in \mathbb{Z}$ (see [3, 7]) and varieties M . For $k \geq 0$, the above formula is equivalent to the statement that $[\text{Sym}^n(\mathbb{C}^k \times M)] = [\mathbb{C}^{nk} \times \text{Sym}^n(M)]$ in $K_0(Var_{\mathbb{C}})$ which is a lemma due to Totaro [6, Lemma 4.4].

The power structure on $K_0(Stck_{\mathbb{C}})$ no longer respects the semi-ring $S_0(Stck_{\mathbb{C}})$ of effective classes, i.e. those spanned by stacks (see [7, § 3]). However, the following partial effectivity result holds:

Lemma 5. *Suppose that M is a variety and A_i are stacks where A_0 is a point. Then the coefficients of $(\sum_{i=0}^{\infty} [A_i] t^i)^{[M]}$ are the classes of stacks, in particular they are given by equation (4) and the geometric interpretation given by equation (5) continues to hold.*

Proof. Let $B(t) = \sum_{n=0}^{\infty} [B_n] t^n$ where B_n is the stack defined by equation (5). We need to show that $A(t)^{[M]} = B(t)$.

It is a formal consequence of the existence of the power structure and equation (6) that exponentiation commutes with the substitution $t \mapsto \mathbb{L}^N t$, i.e.

$$C(\mathbb{L}^N t)^M = (C(t)^M) |_{t \mapsto \mathbb{L}^N t}$$

for any series $C(t)$ beginning with 1 (see [8, Statement 2]).

We fix positive integers d and D . Using the dimension filtration $F_{-D} \subset K_0(\text{Var}_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ we may consider the series $A(t)$ modulo t^d and modulo elements of dimension $\leq -D$. Since A_1, \dots, A_{d-1} are the classes of stacks, there exists a polynomial

$$\tilde{A}(t) = \sum_{i=0}^{d-1} \tilde{A}_i t^i$$

and an integer $N = N(d, D)$ such that

$$\tilde{A}(t) = A(t) \mod (t^d, F_{-D})$$

and such that $\mathbb{L}^{iN} \tilde{A}_i$ is the class of a variety. Since the desired formula holds for series whose coefficients are varieties, it holds for the series $\tilde{A}(\mathbb{L}^N t)$ and hence we have that

$$A(\mathbb{L}^N t)^{[M]} = B(\mathbb{L}^N t) \mod (t^d, F_{-D})$$

and thus by the substitution rule, we have

$$A(t)^{[M]} = B(t) \mod (t^d, F_{-D}).$$

Since the equality holds for arbitrary d and D , it must hold in $K_0(\text{Var}_{\mathbb{C}})[[\mathbb{L}^{-1}]][[t]]$. \square

Power structures can be easily extended to accommodate power series in several or even an infinite number of variables [9, 10]. Let $\mathbf{t} = (t_1, \dots, t_r)$ denote an r -tuple of variables (r could be countably infinite) and let $R[[\mathbf{t}]]$ be the ring of formal power series $\sum_{\mathbf{k}} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ where the sum is over r -tuples $\mathbf{k} = (k_1, \dots, k_r)$ of non-negative integers³. We adopt the multi-index product convention throughout: $\mathbf{t}^{\mathbf{k}} := \prod_j t_j^{k_j}$.

In [9], Gusein-Zade, Luengo, and Melle-Hernandez, extend power structures to multi-variable power series rings and they extend the geometric interpretation given by equations (4) and (5) to the multi-variable case. Namely, if $[A_{\mathbf{k}}]$ is the class of a variety (stack) and $[M]$ is the class of a variety, then the coefficient of $\mathbf{t}^{\mathbf{n}}$ in the series $\left(\sum_{\mathbf{k}} [A_{\mathbf{k}}] \mathbf{t}^{\mathbf{k}}\right)^{[M]}$ is the variety (stack) parameterizing maps

$$\phi : S \rightarrow \bigsqcup_{\mathbf{k} \neq \mathbf{0}} A_{\mathbf{k}}$$

where $S \subset M$ is a finite subset and

$$\sum_{x \in S} \mathbf{i}(\phi(x)) = \mathbf{n}$$

where

$$\mathbf{i} : \sqcup A_{\mathbf{i}} \rightarrow \mathbb{Z}_{\geq 0}^r$$

is the tautological index map. This again can be interpreted as parameterizing “particles” on M of total “charge” \mathbf{n} where “charge” now is given by an r -tuple of integers and the

³If r is infinite, then we require all but a finite number of the k_i s to be zero.

“internal state space” of a charge \underline{k} particle is parameterized by $A_{\underline{k}}$. The proof of Lemma 5 works in this slightly more general setting.

We note that unpacking the definition of the multivariable power structure, we find that

$$(1 - \underline{t}^{\underline{n}})^{-\mathbb{L}^k} = (1 - \mathbb{L}^k \underline{t}^{\underline{n}})^{-1}$$

for each monomial $\underline{t}^{\underline{n}}$.

3. THE MAIN ARGUMENT

In this section, we first use the power structure on $K_0(Stck_{\mathbb{C}})$ to provide a short and easy proof of the Feit-Fine formula (equation (1)). We then refine the basic argument to prove our refinements and generalizations of the Feit-Fine formula.

3.1. Proof of the Feit-Fine formula. We begin with the observation that the motivic class

$$\frac{[C(n)]}{[GL(n)]} \in K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}, (1 - \mathbb{L}^b)^{-1}] \cong K_0(Stck_{\mathbb{C}})$$

has a geometric interpretation as the class of a natural stack. Namely, let

$$\mathrm{Coh}_n(\mathbb{C}^2)$$

denote the stack of coherent sheaves on the affine plane which are supported at points and of length n . This is equivalent to the stack of modules M over the ring $\mathbb{C}[x, y]$ such that $\dim_{\mathbb{C}} M = n$. Given a $\mathbb{C}[x, y]$ module M and a linear isomorphism $M \cong \mathbb{C}^n$, the action of x and y on M yields a pair (A, B) of commuting $n \times n$ matrices. Dividing out by the choice of the isomorphism, this correspondence induces a stack equivalence

$$(7) \quad C(n)/GL(n) \cong \mathrm{Coh}_n(\mathbb{C}^2).$$

Since all $GL(n)$ torsors are trivial (essentially by definition [4]) in $K_0(Stck_{\mathbb{C}})$, the above yields

$$\sum_{n=0}^{\infty} \frac{[C(n)]}{[GL(n)]} t^n = \sum_{n=0}^{\infty} [\mathrm{Coh}_n(\mathbb{C}^2)] t^n$$

in $K_0(Stck_{\mathbb{C}})[[t]]$.

Let

$$\mathrm{Coh}_n^{(0,0)}(\mathbb{C}^2)$$

be the stack of length n coherent sheaves supported at $(0, 0) \in \mathbb{C}^2$. Then the geometric interpretation of the power structure on $K_0(Stck_{\mathbb{C}})$ implies that

$$(8) \quad \sum_{n=0}^{\infty} [\mathrm{Coh}_n(\mathbb{C}^2)] t^n = \left(\sum_{n=0}^{\infty} [\mathrm{Coh}_n^{(0,0)}(\mathbb{C}^2)] t^n \right)^{[\mathbb{C}^2]}.$$

Indeed, since the stack $\mathrm{Coh}_n^{(a,b)}(\mathbb{C}^2)$ of length n sheaves supported at $(a, b) \in \mathbb{C}^2$ is canonically isomorphic to $\mathrm{Coh}_n^{(0,0)}(\mathbb{C}^2)$, we see that as required by equation (5), $\mathrm{Coh}_n(\mathbb{C}^2)$ parameterizes maps

$$\phi : S \rightarrow \bigsqcup_{i=1}^{\infty} \mathrm{Coh}_i^{(0,0)}(\mathbb{C}^2)$$

where $S \subset \mathbb{C}^2$ is the support set of the sheaf and ϕ identifies the sheaf restricted to each point of its support with the corresponding sheaf supported at $(0, 0)$.

Let

$$\mathrm{Coh}_n^{(0,* \neq 0)}(\mathbb{C}^2)$$

be the stack of length n coherent sheaves whose support lies on points of the form $(0, b) \in \mathbb{C}^2$ with $b \neq 0$. Then by an argument similar to above we have

$$(9) \quad \sum_{n=0}^{\infty} [\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2)] t^n = \left(\sum_{n=0}^{\infty} [\mathrm{Coh}_n^{(0, 0)}(\mathbb{C}^2)] t^n \right)^{[\mathbb{C} - \{0\}]}$$

Combining equations (8) and (9) and applying the power structure axioms we get

$$\sum_{n=0}^{\infty} [\mathrm{Coh}_n(\mathbb{C}^2)] t^n = \left(\sum_{n=0}^{\infty} [\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2)] t^n \right)^{\frac{\mathbb{L}^2}{\mathbb{L}-1}}.$$

Note that under the equivalence given by equation (7), a point in the substack $\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2) \subset \mathrm{Coh}_n(\mathbb{C}^2)$ corresponds to a pair of matrices (A, B) where the eigenvalues of A are all 0 and the eigenvalues of B are all non-zero. In other words, $\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2)$ is equivalent to the stack quotient:

$$\{(A, B) \in \mathrm{End}(n)^2 : [A, B] = 0, A \text{ is nilpotent}, B \text{ is invertible}\} / GL(n).$$

For any partition $\lambda \vdash n$, let J_λ denote the unique nilpotent matrix in Jordan normal form having Jordan type λ . Then using the action of $GL(n)$ to put A in Jordan normal form, we get an equivalence

$$\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2) \cong \bigsqcup_{\lambda} \{B \in GL(n) : [J_\lambda, B] = 0\} / \mathrm{Stab}_{J_\lambda}(GL(n))$$

where $\mathrm{Stab}_{J_\lambda}(GL(n)) \subset GL(n)$ is the stabilizer of J_λ under the conjugation action. Since the equation $[J_\lambda, B] = 0$ is equivalent to the equation $B^{-1}J_\lambda B = J_\lambda$, the numerator and the denominator of the above stack quotient are the same. Thus in $K_0(\mathrm{Stck}_{\mathbb{C}})$ we simply have

$$[\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2)] = p(n)$$

where $p(n)$ is the number of partitions of n .

Putting it all together, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[C(n)]}{[GL(n)]} t^n &= \left(\sum_{n=0}^{\infty} [\mathrm{Coh}_n^{(0, * \neq 0)}(\mathbb{C}^2)] t^n \right)^{\frac{\mathbb{L}^2}{\mathbb{L}-1}} \\ &= \left(\sum_{n=0}^{\infty} p(n) t^n \right)^{\mathbb{L} + 1 + \mathbb{L}^{-1} + \mathbb{L}^{-2} + \dots} \\ &= \prod_{k=1}^{\infty} \left(\prod_{m=1}^{\infty} (1 - t^m)^{-1} \right)^{\mathbb{L}^{2-k}} \\ &= \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 - \mathbb{L}^{2-k} t^m)^{-1}. \end{aligned}$$

Here we are working over the completed ring $K_0(\mathrm{Var}_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ and we have used equation (6). The above equation is equivalent by expansion to the Feit-Fine formula and thus completes our proof.

3.2. Proof of Theorem 1/Theorem 1'. This proof is a straightforward generalization of the previous proof utilizing the multi-variable generalization of power structures.

Let $\underline{b} = (b_1, b_2, \dots)$ be a countable tuple of non-negative integers with a finite number of non-zero entries. Let

$$\alpha = (1^{b_1}, 2^{b_2}, \dots)$$

be the corresponding partition of $n = \sum_k k b_k$, i.e. α is the partition having b_k parts of size k .

Let

$$\text{Coh}_{\underline{b}}(\mathbb{C}^2)$$

denote the stack of length n coherent sheaves F on \mathbb{C}^2 such that multiplication by the x coordinate on $H^0(\mathbb{C}^2, F)$ has Jordan type α .

Via the same argument leading to equation (7), we have the equivalence of stacks

$$\frac{C(\alpha)}{GL(n)} \cong \text{Coh}_{\underline{b}}(\mathbb{C}^2).$$

Thus we get the equation

$$\sum_{\underline{b}} \frac{[C(\alpha)]}{[GL(n)]} \underline{t}^{\underline{b}} = \sum_{\underline{b}} [\text{Coh}_{\underline{b}}(\mathbb{C}^2)] \underline{t}^{\underline{b}}$$

in the ring $K_0(\text{Stck}_{\mathbb{C}})[[\underline{t}]]$.

Let $\text{Coh}_{\underline{b}}^{(0,0)}$ and $\text{Coh}_{\underline{b}}^{(0,*\neq 0)}$ be the substacks of $\text{Coh}_{\underline{b}}(\mathbb{C}^2)$ parameterizing sheaves supported at the origin and at points of the form $(0, y)$ with $y \neq 0$ respectively. The geometric interpretation of the (multi-variable) power structure yields

$$\sum_{\underline{b}} [\text{Coh}_{\underline{b}}(\mathbb{C}^2)] \underline{t}^{\underline{b}} = \left(\sum_{\underline{b}} [\text{Coh}_{\underline{b}}^{(0,0)}(\mathbb{C}^2)] \underline{t}^{\underline{b}} \right)^{\mathbb{L}^2}$$

since once again we may use translation to canonically identify $\text{Coh}_{\underline{b}}^{(x_0, y_0)}(\mathbb{C}^2)$ with $\text{Coh}_{\underline{b}}^{(0,0)}(\mathbb{C}^2)$ to see that $\text{Coh}_{\underline{n}}(\mathbb{C}^2)$ parameterizes maps

$$\phi : S \rightarrow \bigsqcup_{\underline{b}} \text{Coh}_{\underline{b}}^{(0,0)}(\mathbb{C}^2)$$

where $S \subset \mathbb{C}^2$ is the support of the sheaf and $\underline{n} = \sum_{x \in S} \underline{b}(\phi(x))$.

Arguing as in § 3.1, we arrive at the equation

$$\sum_{\underline{b}} [\text{Coh}_{\underline{b}}(\mathbb{C}^2)] \underline{t}^{\underline{b}} = \left(\sum_{\underline{b}} [\text{Coh}_{\underline{b}}^{(0,*\neq 0)}(\mathbb{C}^2)] \underline{t}^{\underline{b}} \right)^{\frac{\mathbb{L}^2}{\mathbb{L}-1}}.$$

We then argue as before to find that the motivic class of stack $\text{Coh}_{\underline{\mathbb{L}}}^{(0, * \neq 0)}(\mathbb{C}^2)$ is just 1 so that we get

$$\begin{aligned} \sum_{\underline{\mathbb{L}}} [\text{Coh}_{\underline{\mathbb{L}}}(\mathbb{C}^2)] \underline{\mathbb{L}}^{\underline{\mathbb{L}}} &= \left(\sum_{\underline{\mathbb{L}}} \underline{\mathbb{L}}^{\underline{\mathbb{L}}} \right)^{\mathbb{L}+1+\mathbb{L}^{-1}+\mathbb{L}^{-2}+\dots} \\ &= \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 - t_m)^{-\mathbb{L}^{2-k}} \\ &= \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 - \mathbb{L}^{2-k} t_m)^{-1}. \end{aligned}$$

3.3. Proof of Euler's formula. Applying the argument of § 3.1 to the affine line instead of the affine plane gives us a geometric proof of a more elementary formula due to Euler. Let

$$\text{Coh}_n(\mathbb{C})$$

be the stack of length n coherent sheaves on the affine line. It is equivalent to the stack of $\mathbb{C}[x]$ modules of dimension n and hence

$$\text{Coh}_n(\mathbb{C}) \cong \text{End}(n)/GL(n).$$

Similarly, the stacks of coherent sheaves supported at the origin and away from the origin respectively are given by nilpotent and invertible matrices up to conjugation:

$$\text{Coh}_n^0(\mathbb{C}) \cong \text{Nil}(n)/GL(n), \quad \text{Coh}_n^{* \neq 0}(\mathbb{C}) \cong GL(n)/GL(n).$$

The geometric interpretation of the power structure then implies

$$\begin{aligned} \sum_{n=0}^{\infty} [\text{Coh}_n(\mathbb{C})] t^n &= \left(\sum_{n=0}^{\infty} [\text{Coh}_n^0(\mathbb{C})] t^n \right)^{\mathbb{L}} \\ \sum_{n=0}^{\infty} [\text{Coh}_n^{* \neq 0}(\mathbb{C})] t^n &= \left(\sum_{n=0}^{\infty} [\text{Coh}_n^0(\mathbb{C})] t^n \right)^{\mathbb{L}-1} \end{aligned}$$

Consequently we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[\text{End}(n)]}{[GL(n)]} t^n &= \left(\sum_{n=0}^{\infty} [\text{Coh}_n^{* \neq 0}(\mathbb{C})] t^n \right)^{\frac{\mathbb{L}}{\mathbb{L}-1}} \\ &= \left(\sum_{n=0}^{\infty} t^n \right)^{1+\mathbb{L}^{-1}+\mathbb{L}^{-2}+\dots} \\ &= \prod_{k=0}^{\infty} (1 - t)^{-\mathbb{L}^{-k}} \\ (10) \quad &= \prod_{k=0}^{\infty} (1 - \mathbb{L}^{-k} t)^{-1}. \end{aligned}$$

Since

$$\frac{[\text{End}(n)]}{[GL(n)]} = \frac{\mathbb{L}^{n^2}}{(\mathbb{L}^n - \mathbb{L}^{n-1}) \dots (\mathbb{L}^n - 1)} = \frac{1}{(1 - \mathbb{L}^{-1}) \dots (1 - \mathbb{L}^{-n})}$$

we can set $q = \mathbb{L}^{-1}$ to get the more familiar version of Euler's formula

$$\sum_{n=0}^{\infty} \frac{t^n}{(1-q) \cdots (1-q^n)} = \prod_{k=0}^{\infty} (1 - q^k t)^{-1}.$$

3.4. Proof of Theorem 2. This proof follows a similar script to our previous proofs. The key is to interpret the variety $C(\underline{n})$ in terms of sheaves on an orbifold quotient of \mathbb{C}^2 . Namely, we consider the orbifold given by the following stack quotient

$$\sqrt[r]{\mathbb{C}} \times \mathbb{C} := \mathbb{C}/(\mathbb{Z}/r) \times \mathbb{C}$$

where $k \in \mathbb{Z}/r$ acts on \mathbb{C} by multiplication by $\exp(2\pi i k/r)$.

A coherent sheaf on $\sqrt[r]{\mathbb{C}} \times \mathbb{C}$ may be regarded as a coherent sheaf on \mathbb{C}^2 , invariant under the action of \mathbb{Z}/r . If F is a \mathbb{Z}/r -invariant sheaf on \mathbb{C}^2 , then $H^0(\mathbb{C}^2, F)$ is naturally a \mathbb{Z}/r representation. For each $\underline{n} = (n_1, \dots, n_r)$, let

$$\mathrm{Coh}_{\underline{n}}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})$$

be the stack of sheaves on $\sqrt[r]{\mathbb{C}} \times \mathbb{C}$ such that the dimension of the weight k space in the \mathbb{Z}/r representation $H^0(\mathbb{C}^2, F)$ is n_k .

We fix a \mathbb{Z}/r representation $V = V_1 \oplus \cdots \oplus V_r$ such that V_k , the weight k subrepresentation, has dimension n_k . For any object $F \in \mathrm{Coh}_{\underline{n}}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})$, we fix an isomorphism

$$H^0(\mathbb{C}^2, F) \cong V.$$

Note that under this identification, multiplication by x on V (regarded as an $\mathbb{C}[x, y]$ -module) takes V_i to V_{i+1} and multiplication by y takes V_i to V_i . Dividing out by the choice of the isomorphism, the identification induces a stack equivalence:

$$(11) \quad C(\underline{n})/G(\underline{n}) \cong \mathrm{Coh}_{\underline{n}}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})$$

and hence leads to the equality

$$(12) \quad \sum_{\underline{n}} \frac{[C(\underline{n})]}{[G(\underline{n})]} \underline{t}^{\underline{n}} = \sum_{\underline{n}} [\mathrm{Coh}_{\underline{n}}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{t}^{\underline{n}}$$

in the ring $K_0(\mathrm{Stck}_{\mathbb{C}})[[t_1, \dots, t_r]]$.

To analyze the series $\sum_{\underline{n}} [\mathrm{Coh}_{\underline{n}}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{t}^{\underline{n}}$ using power structures, we must consider the stacky points and non-stacky points of $\sqrt[r]{\mathbb{C}} \times \mathbb{C}$ separately. We use

$$\mathrm{Coh}_{\underline{n}}^{(0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C}), \text{ and } \mathrm{Coh}_{\underline{n}}^{(*\neq 0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})$$

respectively to denote the substack of sheaves supported on respectively the stacky locus $B\mathbb{Z}/r \times \mathbb{C} \subset \sqrt[r]{\mathbb{C}} \times \mathbb{C}$ and its complement $\mathbb{C}^*/(\mathbb{Z}/r) \times \mathbb{C} \subset \sqrt[r]{\mathbb{C}} \times \mathbb{C}$.

We note that $\mathrm{Coh}_{\underline{n}}^{(*\neq 0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})$ is empty unless $n_1 = \cdots = n_r$ in which case we have an equivalence

$$\mathrm{Coh}_{(n, \dots, n)}^{(*\neq 0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C}) \cong \mathrm{Coh}_n^{(*\neq 0,*)}(\mathbb{C}^2)$$

induced by the equivalence $\mathbb{C}^*/(\mathbb{Z}/r) \times \mathbb{C} \cong \mathbb{C}^* \times \mathbb{C}$. Letting

$$t = t_1 \cdots t_r$$

and using the power structure as in § 3.1, we get

$$\begin{aligned}
 \sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(*\neq 0, *)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{\mathbf{t}}^{\underline{n}} &= \sum_{n=0}^{\infty} [\text{Coh}_n^{(*\neq 0, *)}(\mathbb{C}^2)] t^n \\
 &= \left(\sum_{n=0}^{\infty} [\text{Coh}_n^{(*\neq 0, 0)}(\mathbb{C}^2)] t^n \right)^{\mathbb{L}} \\
 &= \left(\sum_{n=0}^{\infty} p(n) t^n \right)^{\mathbb{L}} \\
 &= \left(\prod_{m=1}^{\infty} (1 - t^m)^{-1} \right)^{\mathbb{L}} \\
 (13) \quad &= \prod_{m=1}^{\infty} (1 - \mathbb{L} t^m)^{-1}.
 \end{aligned}$$

Using the geometric interpretation of the multi-variable version of the power structure, we may use arguments similar to the ones the previous proofs to get

$$\begin{aligned}
 \sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(0, *)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{\mathbf{t}}^{\underline{n}} &= \left(\sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(0, 0)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{\mathbf{t}}^{\underline{n}} \right)^{\mathbb{L}} \\
 (14) \quad &= \left(\sum_n [\text{Coh}_{\underline{n}}^{(0, *\neq 0)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{\mathbf{t}}^{\underline{n}} \right)^{\frac{\mathbb{L}}{\mathbb{L}-1}}.
 \end{aligned}$$

The right hand side in the above equation will be determined using the following lemma.

Lemma 6 (c.f. Lemma 4.8 of [12]). *The class $[\text{Coh}_{\underline{n}}^{(0, *\neq 0)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \in K_0(\text{Stck}_{\mathbb{C}})$ is the positive integer given by the number of collections $\{\lambda(a, b)\}$ consisting of partitions indexed by $(a, b) \in (\mathbb{Z}/r)^2$ satisfying*

$$(15) \quad n_j = \sum_{(a, b)} \left\{ |\lambda(a, b)| - \sum_{[a, b] \not\ni j} l(\lambda(a, b)) \right\}$$

where $l(\lambda(a, b))$ denotes the length of the partition $\lambda(a, b)$ and $[a, b] \subset \mathbb{Z}/r$ is the “interval” $a, a+1, \dots, b$.

Proof. Recall that an endomorphism A of $V_1 \oplus \dots \oplus V_r$ is of *cyclic block type* if $A = A_1 \oplus \dots \oplus A_r$ where $A_i \in \text{Hom}(V_i, V_{i+1})$ and B is of *diagonal block type* if $B = B_1 \oplus \dots \oplus B_r$ where $B_i \in \text{Hom}(V_i, V_i)$. Note that B is invertible and of diagonal block type if and only if $B \in G(\underline{n})$.

The stack equivalence (11) then induces an equivalence between $\text{Coh}_{\underline{n}}^{(0, *\neq 0)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})$ and the stack quotient

$$\{A, B \in \text{End}(V) : [A, B] = 0, A \text{ is nilpotent of cyclic block type}, B \in G(\underline{n})\} / G(\underline{n}).$$

We now analyze Jordan normal form for nilpotent endomorphisms of cyclic block type.

We suppose that A is of cyclic block type and is nilpotent. Jordan canonical form says there exists vectors $e_1, \dots, e_l \in V$ such that the collection of non-zero vectors of the form $A^j e_i$ is a basis of V . Moreover, without loss of generality, we may assume that each e_i

(and hence each $A^j e_i$) lies in a single summand of $V = V_1 \oplus \cdots \oplus V_r$. We will say that a vector $f \in \{e_1, \dots, e_l\}$ *starts at a and ends at b* if $f \in V_a$ and $A^k f \in V_b$ where k is the largest integer such that $A^k f \neq 0$. We define the length of f to be $\lfloor \frac{k}{r} \rfloor + 1$. We define a collection of partitions $\{\lambda(a, b)\}$ by declaring that the number of parts of size j in the partition $\lambda(a, b)$ is the number of vectors in the collection $\{e_1, \dots, e_l\}$ which start at a , end at b , and have length j .

Note that the dimensions of V_j can be written in terms the partitions $\lambda(a, b)$ and they are given precisely by equation (15).

The collection $\{\lambda(a, b)\}$ uniquely determines a nilpotent endomorphism of cyclic block type up to conjugation by elements in $G(\underline{n})$.

For each collection partitions $\{\lambda(a, b)\}$, let $J_{\{\lambda(a, b)\}}$ be the matrix in the Jordan form described above. Then we have

$$\begin{aligned} & \{A, B \in \text{End}(V) : [A, B] = 0, A \text{ is nilpotent of cyclic block type}, B \in G(\underline{n})\} / G(\underline{n}) \\ & \cong \bigsqcup_{\{\lambda(a, b)\}} \{B \in G(\underline{n}) : [J_{\{\lambda(a, b)\}}, B] = 0\} / \text{Stab}_{J_{\{\lambda(a, b)\}}}(G(\underline{n})) \end{aligned}$$

where the union is taken over all collections $\{\lambda(a, b)\}$ satisfying equation (15) and where

$$\text{Stab}_{J_{\{\lambda(a, b)\}}}(G(\underline{n})) \subset G(\underline{n})$$

is the stabilizer of $J_{\{\lambda(a, b)\}}$ under the action of conjugation by elements of $G(\underline{n})$.

Since the equation $[J_{\{\lambda(a, b)\}}, B] = 0$ is equivalent to the equation $B J_{\{\lambda(a, b)\}} B^{-1} = J_{\{\lambda(a, b)\}}$, the numerator and the denominator of the above stack quotient are the same. Therefore, in the Grothendieck group, each factor in the disjoint union contributes 1. The lemma follows. \square

We can now use the lemma to compute. In the below, $\{\lambda(a, b)\}$ always denotes a collection of partitions indexed by $(a, b) \in (\mathbb{Z}/r)^2$ and we use the notation $]a, b[$ to denote the complement of the “interval” $[a, b] = \{a, a+1, \dots, b\} \subset \mathbb{Z}/r$ (so for example, $]a, b[$ is empty if $b = a-1$). For a subset $S \subset \mathbb{Z}/r$, let t_S denote the product $\prod_{s \in S} t_s$. Applying the lemma we get

$$\begin{aligned} \sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(0, * \neq 0)}(\sqrt{r}\mathbb{C} \times \mathbb{C})] \underline{t}^{\underline{n}} &= \sum_{\{\lambda(a, b)\}} (t_1 \cdots t_r)^{|\lambda(a, b)|} \prod_{j \notin [a, b]} t_j^{-l(\lambda(a, b))} \\ &= \prod_{(a, b)} \left(\sum_{\lambda(a, b)} t^{|\lambda(a, b)|} \cdot t_{]a, b[}^{-l(\lambda(a, b))} \right) \\ &= \prod_{(a, b)} \prod_{m=1}^{\infty} (1 - t_{]a, b[}^{-1} t^m)^{-1} \\ &= \prod_{(a, b)} \prod_{m=1}^{\infty} (1 - t_{[a, b]} t^{m-1})^{-1}. \end{aligned}$$

The equality from the second to the third line in the above follows from the well-known formula

$$\sum_{\lambda} u^{|\lambda|} v^{l(\lambda)} = \prod_{m=1}^{\infty} (1 - v u^m)^{-1}.$$

Combining the above with equation (14) we get

$$\begin{aligned}
 \sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{t}^{\underline{n}} &= \left(\prod_{(a,b)} \prod_{m=1}^{\infty} (1 - t_{[a,b]} t^{m-1})^{-1} \right)^{\frac{\mathbb{L}}{\mathbb{L}-1}} \\
 &= \prod_{k=0}^{\infty} \prod_{(a,b)} \prod_{m=1}^{\infty} (1 - t_{[a,b]} t^{m-1})^{-\mathbb{L}^{-k}} \\
 (16) \quad &= \prod_{m=1}^{\infty} \prod_{k=0}^{\infty} \prod_{(a,b)} (1 - \mathbb{L}^{-k} t_{[a,b]} t^{m-1})^{-1}.
 \end{aligned}$$

Finally, a coherent sheaf on $\sqrt[r]{\mathbb{C}} \times \mathbb{C}$ is the direct sum of a sheaf supported on $\mathbb{C}^*/(\mathbb{Z}/r) \times \mathbb{C}$ with a sheaf supported on $B\mathbb{Z}/r \times \mathbb{C}$, and the numerical invariants \underline{n} are additive. Consequently we get the equation

$$\sum_{\underline{n}} [\text{Coh}_{\underline{n}}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{t}^{\underline{n}} = \left(\sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(* \neq 0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{t}^{\underline{n}} \right) \left(\sum_{\underline{n}} [\text{Coh}_{\underline{n}}^{(0,*)}(\sqrt[r]{\mathbb{C}} \times \mathbb{C})] \underline{t}^{\underline{n}} \right)$$

Substituting equations (12), (13), and (16) into the above equation yields

$$\sum_{\underline{n}} \frac{[C(\underline{n})]}{[G(\underline{n})]} \underline{t}^{\underline{n}} = \left(\prod_{m=1}^{\infty} (1 - \mathbb{L} t^m)^{-1} \right) \left(\prod_{m=1}^{\infty} \prod_{k=0}^{\infty} \prod_{(a,b)} (1 - \mathbb{L}^{-k} t_{[a,b]} t^{m-1})^{-1} \right)$$

which is easily rewritten as the equation in Theorem 2 and thus completes its proof.

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